PMM U.S.S.R.,Vol.45,pp. 636-645 0021-8928/82/5 0636 $7.50/0

UDC 539.3

GENERAL TECHNIQUE FOR THE SOLUTION OF NONHOMOGENEOUS LINEAR PROBLEMS FOR SYMMETRIC MECHANICAL SYSTEMS*

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A technique for simplifying the computation of the stress-deformed state of a linear symmetric mechanical system affected by nonsymmetric loads is studied. A nonhomogeneous equation formulated in general form encompasses a variety of problems in the mechanics of deformed body. The statement of the problem, individual results, and stages of the proposed technique are illustrated by discrete methods and the two-dimensional problem of the theory of elasticity.

1. Abstract statement. Let us consider an elasto-linear mechanical system S in a region Ω and formulate a nonhomogeneous problem for it which reduces to the solution of the operator equation

$$Au = v, \quad u \in L_1, \quad v \in L_2 \tag{1.1}$$

Here L_1 and L_2 are given spaces of functions defined on Ω, and A is a linear operator defined from L_1 into L_2. The function u describes the stress-deformed state of the system S, while the function v is the specified loads and displacements. For the sake of brevity, u and v will be called the state and load functions, respectively.

Usually, equation (1.1) is not studied directly, but is instead replaced by a formally defined resolvent equation

$$BU = V, \ U \in L'_1, \ V \in L_2' \tag{1.2}$$

where L_1' and L_2' are spaces of formalized state and load functions, and B is a linear operator defined from L_1' into L_2'. Here we have the relations

$$u = B_1 U, \quad V = B_2 v \tag{1.3}$$

We will understand by B_1 and B_2 well-known linear operators defined from L_1' into L_1 and L_2 into L_2'. Since elements from the null-spaces of the operators A, B_1 and B_2 are not of interest in most problems in the mechanics of deformed bodies, their selection will henceforth depend upon considerations of compactness in the presentation.

Let us consider three examples of this formalization which are widespread and which we will study below.

Discrete methods. In place of Ω, we will use a discrete (net-point) region. The operator B_2 replaces the load by forces concentrated at the nodes of the net, and replace the function v by the vector V whose components describe the specified forces and variables at the nodes.

The operator B_1 is responsible for a gradual transition from the components of the vector U to values of the components of the stress-deformed state at the network nodes and their interpolation at other points of the region Ω.

The finite-dimensional spaces L_1' and L_2' merge into a single space L'. The selection of the coordinate axes and the unknowns by the calculator essentially defines a basis in L'. The operator B is replaced by a matrix corresponding to it in the spaces and the resolvent equation (1.2), by a system of linear algebraic equations.

Plane problem of elasticity theory for an isotropic medium /1/. The plane in which the region Ω is located is assumed to be the complex plane, and $z = x + iy$ is the affix of points with coordinates x and y. In place of (1.2) we have

$$K_1\varphi(t_p) + K_2 \overline{[t_p\varphi'(t_p) + \psi(t_p)]} = f_p(t_p) \quad (p = 1, 2, \ldots, N_0) \tag{1.4}$$

*Prikl.Matem.Mekhan.,45,No.5,849-861,1981 where K_1 and K_2 are coefficients that depend upon the boundary conditions; N_0 is quantity of paths forming the boundary of Ω, p is number of path; t_p is a point on p-th path; and $\varphi(z)$ and $\psi(z)$ are the Kolosov–Muskhelishvili complex potentials analytic in Ω. The abstract concepts introduced above here assume the following meaning: $U = \{\varphi(z), \psi(z)\}$, $V = \{f_p(t_p)\}_{p=1}^{N_0}$. The operators B, B_1 and B_2 are described by equations (1.4), the relation between the complex potentials and the components of the stress-deformed state, and the relation between the load and functions $f_p(t_p)$, respectively.

Plane problem for anisotropic medium /2,3/. The formalization applied here is similar to the above. The equations

$$2\mathrm{Re}\,[K_{1r}W_1(t_{p1}) + K_{2r}W_2(t_{p2})] = f_{pr}(t_p) \quad (p = 1, 2, \ldots, N_0;\ r = 1, 2) \tag{1.5}$$

are the resolvent equations, where K_{jr} $(j, r = 1, 2)$ are coefficients that depend upon the boundary conditions

$$z_j = x + \mu_j y \quad (j = 1, 2) \tag{1.6}$$

$W_j(z_j)$ are the Lekhnitskii complex potentials which are analytic in the region Ω_j, and both Ω_j and t_{pj} are obtained from Ω and t_p by the transformation (1.6). Here

$$U = \{W_1(z_1),\ W_2(z_2)\},\quad V = \{f_{p1}, f_{p2}\}_{p=1}^{N_0}$$

2. Symmetry of mechanical systems /4/. A motion in space in which a system S reaches a position indistinguishable from its initials position is said to be an element of symmetry of the system. The elements include a turn C_m $(m = 0, 1, \ldots, n-1)$ about an n-th order axis, a reflection Θ in some plane, a translation (parallel motion) $T_{m_1m_2m_3}$ $(m_1, m_2, m_3 = 0, \pm1, \pm2, \ldots)$ by a vector $m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3$, where $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ are the basic vectors, a trivial motion $e = C_0 = T_{000}$ and so on. For these elements of symmetry, we introduce the operation of multiplication, taking their composition as a product. Then the set G of elements of symmetry is a group.

A mechanical system S with symmetry group G may be partitioned into identical parts, or elementary cells, such that because of the effect of some nontrivial element $g \in G$, each of the cells is made to move, and combine with some other cell. These parts may be conveniently "enumerated" by means of the symmetry elements. We fix some cell S^e and call it the fundamental cell, while the cell obtained from S^e by the motion $g \in G$ is denoted S^g, i.e., $S^g = gS^e$. We partition this region Ω into cells Ω^g $(g \in G)$ assuming that cell S^g is in Ω^g.

The values of the components of the stress-deformed state and the loads described for the functions u and v depend upon the system of coordinate axes. In the case of a symmetric construction, we will use an invariant system, which is understood to consist in the introduction into each cell Ω^g of Ω a local frame of reference $\omega^g = g\omega^e$ obtained from ω^e of the fundamental cell by the motion g.

We alter the construction S by an element of symmetry g. In this case the stress-deformed state at any fixed point of the construction remains invariant, though the point it self moves relative to region Ω. We let z and gz denote points in Ω with which the given point of the construction coincides with before and after the motion. The stress-deformed state of the construction gS is described by the function u_g whose values at the point $gz \in \Omega$ is equal (in an invariant frame of reference) to the value of u at a point z, i.e., $u_g(gz) = u(z)$ or $u_g(z) = u(g^{-1}z)$ where g^{-1} is the motion inverse to the element g.

We introduce an abstract rule for the effect of the element $g \in G$ on any function F defined on Ω, assuming that the function $gF = F_g$ as a result of this operation, such that

$$F_g(z) = F(g^{-1}z) \tag{2.1}$$

Let us stress the mechanical interpretation of rule (2.1): gu and gv are functions of the state and load in the construction gS. Such an interpretation is convenient for explaining the basic properties of a nonhomogeneous problem caused by the symmetry of the mechanical system S.

Property 1. If $u \in L_1$ $(v \in L_2)$, then $gu \in L_1$ $(gv \in L_2)$.

In fact, the spaces L_1 and gL_1 of the state functions S and gS coincide because of the "indistinguishability" of the latter. Moreover, the equation $Au_g = v_g$ is satisfied, along with (1.1). Hence:

Property 2. $Ag = gA$.

3. Generalized symmetry problems. The symmetry properties functions defined on a region Ω with symmetry group G are highly varied. They may be described with maximal completeness and meaningfulness by means of the special apparatus of irreducible representations of groups /4/.

By an $m_{k\nu}$-dimensional irreducible representation $\tau_{k\nu}$ of a group G we will understand a set of given unitary matrices $\tau_{k\nu}(g)$ $(g \in G)$ of order $m_{k\nu}$ that possess a number of special properties. Any symmetry group has its own, known set of irreducible representations defined in advance. The irreducible representations $\tau_{k\nu}$ of a group G may be distinguished by two indices: a vector index k and a scalar index ν, and $k = 0$ for groups with a finite number of elements. The identity representation only τ_{01}, such that $m_{01} = 1$ and $\tau_{01}(g) \equiv 1$, is the simplest of the irreducible representations.

Every representation $\tau_{k\nu}$ describes the symmetry properties of some set consisting of $m_{k\nu}$ functions $F_{k\nu\rho}$ $(\rho = 1, 2, \ldots m_{k\nu})$ which may be transformed according to this representation. The latter signifies that any (μ-th) function in the set satisfies the conditions

$$gF_{k\nu\mu} = \sum_{\rho=1}^{m_{k\nu}} \tau_{k\nu\rho\mu}(g) F_{k\nu\rho}, \quad \forall g \in G \tag{3.1}$$

where $\tau_{k\nu\rho\mu}(g)$ is the $\rho\mu$-th element of the matrix $\tau_{k\nu}(g)$.

We write down the property (3.1) at some point $z \in \Omega^e$, and also bear in mind equality (2.1) and the fact that the matrices $\tau_{k\nu}(g)$ are unitary, thereby obtaining

$$F_{k\nu\mu}(gz) = \sum_{\rho=1}^{m_{k\nu}} \overline{\tau_{k\nu\mu\rho}(g)} F_{k\nu\rho}(z), \quad \forall g \in G, \quad \forall z \in \Omega^e \tag{3.2}$$

From (3.2) it follows that the function $F_{k\nu\mu}$ is uniquely defined by classifying all the functions $F_{k\nu\rho}$ $(\rho = 1, 2, \ldots, m_{k\nu})$ on the cell Ω^e of region Ω.

If the load function in (1.1) occurs in the set transformed by an irreducible representation of group G, the corresponding nonhomogeneous problem will be called a generalized symmetry problem. By condition (3.1), its solution will always be accompanied by simplifications whose nature will be discussed below. Since a symmetric (cyclic, periodic, etc.) load is transformed by the representation τ_{01}, the ordinary symmetry problem is a particular case of the generalized problem.

4. Simplifications in the generalized symmetric problem. We continue the list of symmetry properties of nonhomogeneous problem.

Property 3. A state function may be transformed by a representation $\tau_{k\nu}$ of a group G as the μ-th function in a set,it is necessary and sufficient that the load function is transformed in the same way.

Let us prove, for example, necessity. For this purpose, we assume that the function $u_{k\nu\mu}$ occurs in the set of functions $u_{k\nu\rho}$ $(\rho = 1, 2, \ldots, m_{k\nu})$, transformed by the representation $\tau_{k\nu}$, and introduce the function $v_\rho = Au_{k\nu\rho}$. We operate with the element $g \in G$ on both sides of the equality $v_\mu = Au_{k\nu\mu}$ and also bear in mind property 2 and relation (3.1), obtaining the required result:

$$gv_\mu = gAu_{k\nu\mu} = Agu_{k\nu\mu} = A\sum_{\rho=1}^{m_{k\nu}} \tau_{k\nu\rho\mu}(g)\, u_{k\nu\rho} = \sum_{\rho=1}^{m_{k\nu}} \tau_{k\nu\rho\mu}(g)\, v_\rho$$

Property 4. If the formalized functions $U_{k\nu\rho}$ $(V_{k\nu\rho})$ corresponding the functions $u_{k\nu\rho}$ $(v_{k\nu\rho})$ are written in an invariant frame of reference, they also can be transformed by the representation $\tau_{k\nu}$.

Because of our mechanical interpretation of an effect of a motion g on a function, $gv_{k\nu\mu}$ and $gV_{k\nu\mu}$ must be understood as the ordinary formalized load function of the construction gS. Consequently, $gV_{k\nu\mu} = B_2(gv_{k\nu\mu})$, and, based on (3.1), we have

$$gV_{k\nu\mu} = B_2(gv_{k\nu\mu}) = B_2 \sum_{\rho=1}^{m_{k\nu}} \tau_{k\nu\rho\mu}(g)\, v_{k\nu\rho} = \sum_{\rho=1}^{m_{k\nu}} \tau_{k\nu\rho\mu}(g)\, V_{k\nu\rho}$$

In proving the remaining parts of properties 3 and 4, arbitrary functions of the null-spaces of the operators A and B_1 must be set equal zero.

Property 5. All possible formalized functions $U_{k\nu\mu}$ $(V_{k\nu\mu})$ from the subspace $L'_{1k\nu\mu} \subset L_1'$ $(L'_{2k\nu\mu} \subset L_2')$.

This property, which follows from the linearity of the operators g, B_1 and B_2, may be used to establish the isomorphic correspondences

$$U_{k\nu\mu} \leftrightarrow U^{(k\nu)}, \quad V_{k\nu\mu} \leftrightarrow V^{(k\nu)} \tag{4.1}$$

between the μ-th state (load) functions transformed by the representation $\tau_{k\nu}$ and the elements $U^{(k\nu)}$ $(V^{(k\nu)})$ of the given spaces $L_1^{(k\nu)}$ $(L_2^{(k\nu)})$.

Basic isomorphism. If F is a function defined on region Ω, we will understand by $F|_e$ the function defined on the cell Ω^e by means of the equality

$$F|_e(z) = F(z), \ \forall z \in \Omega^e \tag{4.2}$$

We consider the spaces $L_1^{(k\nu)}$ and $L_2^{(k\nu)}$ formed by all possible sets

$$U^{(k\nu)} = \{U_{k\nu\rho}|_e\}_{\rho=1}^{m_{k\nu}}, \quad V^{(k\nu)} = \{V_{k\nu\rho}|_e\}_{\rho=1}^{m_{k\nu}} \tag{4.3}$$

By virtue of expressions (3.2) and property 4, these sets are isomorphic to the subspaces $L'_{1k\nu\mu}$ and $L'_{2k\nu\mu}$. Thus correspondence (4.1) are established between the elements that satisfy condition (4.3).

In the case of multiply connected regions Ω, it is sometimes convenient to use a modification of the basic isomorphism. This isomorphism essentially indicates that a formalized state function in a number of generalized symmetry problems may be expressed in terms of the sets $U^{(\eta)}$ $(\eta = 1, 2, \ldots, m_{k\nu})$ of certain functions defined on the exterior of the basic bounding surface (contour). Then the space with elements

$$U^{(k\nu)} = \{U^{(\eta)}\}_{\eta=1}^{m_{k\nu}}$$

may be taken as the $L^{(k\nu)}$.

Isotropic medium. We will understand by $U^{(\eta)}$ the set of the two functions $\Phi^{(\eta)}(z)$ and $\Psi^{(\eta)}(z)$ analytic on the exterior of the basic contour. A one-to-one relation between the Kolosov–Muskhelishvili potentials of the generalized symmetry problem and the functions $\Phi^{(\eta)}(z)$, $\Psi^{(\eta)}(z)$ $(\eta = 1, 2, \ldots, m_{k\nu})$ are established by special relations /5/.

Orthotropic medium. Using a previously presented method /5/ and property 3, the complex Lekhnitskii potentials in the generalized symmetry problem may be expressed by means of functions $W_j(z_j)$ analytic on the exterior of the basic contour of region Ω_j:

$$W_{jk\nu\mu}(z_j) = \lim_{N\to\infty} \sum_{m_1=0}^{N} \sum_{m_2=0}^{N} \sum_{m=0}^{1} \sum_{\eta=1}^{m_{k\nu}} (-1)^m \{\tau_{k\nu\mu\rho}(T_{m_1m_2}C_m) \times \tag{4.4}$$

$$W_j^{(\eta)}[(-1)^m (z_j - A_{m_1m_2j})] + \tau_{k\nu\mu\rho}(T_{m_1m_2}C_m\theta)\overline{W_i^{(\eta)}}[(-1)^m (z_j - A_{m_1m_2j})]\}$$

$$A_{m_1m_2j} = m_1a_{11} + m_2a_{21} + \mu_j(m_1a_{12} + m_2a_{22}) \ (\mu = 1, 2, \ldots, m_{k\nu}; \ j = 1, 2)$$

where C_m $(m = 0, 1)$ is a rotation about the origin by an angle $m\pi$; θ, a reflection about the axis x; μ_j $(j = 1, 2)$ the Lekhnitskii complex parameters; and a_{r1} and a_{r2} $(r = 1, 2)$ projections of the vector a_r onto the x- and y-axes. In the right side of relations (4.4) we have retained terms that correspond to elements $g \in G$.

Obviously $U^{(\eta)} = \{W_1^{(\eta)}(z_1), \ W_2^{(\eta)}(z_2)\}$.

Property 6. Under the fixed isomorphisms (4.1), the generalized symmetry problem reduces to the solution of the equation

$$B^{(k\nu)}U^{(k\nu)} = V^{(k\nu)} \tag{4.5}$$

where the operator $B^{(k\nu)}$ is determined by the correspondence

$$B^{(k\nu)}U^{(k\nu)} \leftrightarrow BU_{k\nu\mu} \ (U^{(k\nu)} \leftrightarrow U_{k\nu\mu}) \tag{4.6}$$

The transition from equation (1.2) to (2.5) constitutes an abstract introduction of simplifications into the solution of the generalized symmetry problem.

Discrete methods. We will use the basic isomorphism as condition (4.1). The dimension of the finite-dimensional space $L^{(k\nu)} = L_1^{(k\nu)} = L_2^{(k\nu)}$ is much less than the dimension of L'. Thus, equation (4.5) reduces to a system of low-order algebraic equations. The matrix of this system corresponding to the operator $B^{(k\nu)}$ is constructed in Sect.6.

Plane problem for isotropic medium. We apply the basic isomorphism to the space $L'_{2k\nu\mu}$ i.e., we set $V^{(k\nu)} = \{f^{(\rho)}(t)\}_{\rho=1}^{m_{k\nu}}$, where $t = t_1$ is a point on the basic contour, and $f^{(\rho)}(t)$ is the right side of the corresponding equation (1.4) written for the load $v_{k\nu\rho}$. The required potentials $\varphi_{k\nu\mu}(z)$ and $\psi_{k\nu\mu}(z)$, which are uniquely defined by the functions $f^{(\rho)}(t)$ may be found from the boundary conditions imposed on the basic contour, generated for each of the loads $v_{k\nu\rho}$

$$K_1\varphi_{k\nu\rho}(t) + K_2[t\overline{\varphi'_{k\nu\rho}(t)} + \overline{\psi_{k\nu\rho}(t)}] = f^{(\rho)}(t) \quad (\rho = 1, 2, \ldots, m_{k\nu}) \tag{4.7}$$

We apply the modified isomorphism to the formalized state functions and substitute preious expressions /5/ that relate the potentials $\varphi_{k\nu\rho}(z)$ and $\psi_{k\nu\rho}(z)$ to the functions $\Phi^{(\eta)}(z)$ and $\Psi^{(\eta)}(z)$ $(\eta = 1, 2, \ldots, m_{k\nu})$, in (4.7), obtaining a system of equations relative to $\Phi^{(\eta)}(z)$ and $\Psi^{(\eta)}(z)$. This system is a concrete form of equation (4.5) in our problem and describes the corresponding operator $B^{(k\nu)}$.

Plane problem for orthotropic medium. The construction of the operator $B^{(k\nu)}$, i.e., the system of equations that determines the functions $W_j^{(\eta)}(z_j)$ $(\eta = 1, 2, \ldots, m_{k\nu};\ j = 1, 2)$ in terms of the functions $f_r^{(\rho)}(t)$ $(\rho = 1, 2, \ldots, m_{k\nu};\ r = 1, 2)$, is realized by substituting expressions (4.4) in the equations

$$2\mathrm{Re}\,[K_{1r}W_{1k\nu\rho}(t_{11}) + K_{2r}W_{2k\nu\rho}(t_{12})] = f_r^{(\rho)}(t) \quad (\rho = 1, 2, \ldots, m_{k\nu};\quad r = 1, 2) \tag{4.8}$$

Remark. In the systems (4.7) and (4.8), we have used the boundary conditions only on the basic contour Γ. Let us now consider the contour $\Gamma^g = g\Gamma^e$. For this purpose, the boundary conditions in the generalized symmetry problem have the form

$$(BU_{k\nu\mu})(t^g) = V_{k\nu\mu}(t^g), \quad \forall g \in G, \quad \forall t^g \in \Gamma^g \tag{4.9}$$

Since $BU_{k\nu\mu} \in L_{2k\nu\mu}$, using property 4, equality (3.2) and the self-evident relation $t^g = gt$, we obtain expression (4.9) in the form

$$\sum_{\rho=1}^{m_{k\nu}} \overline{\tau_{k\nu\mu\rho}(g)}\,(BU_{k\nu\rho})(t) = \sum_{\rho=1}^{m_{k\nu}} \overline{\tau_{k\nu\mu\rho}(g)}\,V_{k\nu\rho}(t) \tag{4.10}$$

The equations $(BU_{k\nu\rho})(t) = V_{k\nu\rho}(t)$ $(\rho = 1, 2, \ldots, m_{k\nu})$, constitute boundary conditions of the type (4.7) and (4.8), so that after they are solved, equalities (4.10) and, consdequently, the boundary condition (4.9), will be satisfied automatically.

5. General technique. These simplifications of generalized symmetry problems may be used also for arbitrary loading of a symmetric mechanical system. In fact, practically any load in the construction S with symmetry group G may be represented in the form of a combination of components that may be transformed by means of irreducible representations of group G /6,7/, and because of linearity, the initial problem may be decomposed into several generalized symmetry problems.

This technique of studying a nonhomogeneous problem for a symmetric mechanical system consists in three stages: (a) decomposition of the load into components that may be transformed by means of irreducible representations of the symmetry group; (b) solution of equations (4.5) for the corresponding generalized symmetry problems; (c) superposition of the obtained results.

If we wish to apply this technique to any new class of problems, it becomes necessary to investigate the structure of the spaces $L'_{1k\nu\mu}$ and $L'_{2k\nu\mu}$ and to construct the particular form of the operator $B^{(k\nu)}$. The solutions presented above of such problems for the plane problem of elasticity theory of isotropic and orthotropic media may serve as illustration here. The principle used to construct the operator $B^{(k\nu)}$ is also convenient for many other problems in the theory of thin and thick, densely perforated plates, shells, etc. A specific approach to the solution of these problems for the case of discrete methods may be found in Sect.6.

In those classes of problems for which necessary isomorphisms and the particular form of the operator $B^{(k\nu)}$ have already been established, our technique may be used to greatly reduce the number of interdependent resolvent equations and, consequently, the volume of the computations. A decrease in the number of equations can be seen in all the examples we have presented, bearing in mind that the quantity $m_{k\nu}$ is very small by comparison with N_0, the number of cells in the mechanical system. In these examples, we may verify the following estimate of the efficiency of our computation technique: the volume of computations necessary for solving a generalized symmetric problem may be reduced more than $(N_0/m_{k\nu})^2$ times.

As a rule, the solution of equation (4.5) may be carried out by methods suitable for an ordinary symmetric loading.

6. Discrete methods. It is necessary that the network possess the same symmetry group G as the construction S; we enumerate the nodes of the cell Ω^e from 1 to N (number of nodes in cell). If n_p is the number of degrees of freedom of node $z_p \in \Omega^e$ $(p = 1, 2, \ldots, N)$, then $U\langle t, p, g\rangle$ $(t = 1, 2, \ldots, n_p)$ is the t-th component of the vector U at the node $gz_p \in \Omega^g$. We shall assume that the components $U\langle t, p, e\rangle$ and $U\langle t, p, g\rangle$ have the same physical meaning relative in the frames of reference ω^e and ω^g, respectively, and set

$$U = \| U\langle g\rangle \|_{g\in G}, \quad U\langle g\rangle = \| U\langle p, g\rangle \|_{p=1}^{N}, \quad U\langle p, g\rangle = \| U\langle t, p, g\rangle \|_{t=1}^{n_p} \tag{6.1}$$

Note that the node z_p may, in general, simultaneously occur in several cells. Despite the fact that the number of components in this case is redundant, the structure (6.1) remains highly convenient.

Since the construction S is symmetric, the operator B may be completely determined by the system of algebraic equations

$$\sum_{g\in G} B\langle e, g\rangle U\langle g\rangle = V\langle e\rangle, \quad B\langle e, g\rangle = \| B\langle p, e, q, g\rangle \|_{p,q=1}^{N} \tag{6.2}$$

where $B\langle p, e, q, g\rangle$ are given square matrices of dimension $n_p \times n_q$; columns corresponding to redundant components may be assumed to be empty.

As we noted earlier, the basic isomorphism should be used as the correspondence (4.1). By formulas (4.3), we set

$$\begin{gathered} U^{(\mathbf{k}\nu)} = \| U^{(\mathbf{k}\nu)}\langle p\rangle \|_{p=1}^{N}, \quad U^{(\mathbf{k}\nu)}\langle p\rangle = \| U^{(\mathbf{k}\nu)}\langle \rho, p\rangle \|_{\rho=1}^{m_{\mathbf{k}\nu}} \\ U^{(\mathbf{k}\nu)}\langle \rho, p\rangle = U_{\mathbf{k}\nu\rho}\langle p, e\rangle \end{gathered} \tag{6.3}$$

Let us consider the construction of the basis in the space $L^{(\mathbf{k}\nu)}\langle p\rangle$, formed by the subvectors $U^{(\mathbf{k}\nu)}\langle p\rangle$. For this purpose, we assume that $G^p \subset G$ is the group of node z_p, i.e., the set of elements $g^p \in G$ that leave the node z_p unchanged. Since this node must belong simultaneously to all cells Ω^g $(g \in G^p)$, the following type of relations hold between the components of U:

$$\sum_{q=1}^{n_p} h_{qt}^{(p)}(g^p) U\langle q, p, e\rangle = \sum_{q=1}^{n_p} h_{qt}^{(p)'}(g^p) U\langle q, p, g^p\rangle \quad (t = 1, 2, \ldots, n_p) \tag{6.4}$$

where $h_{qt}^{(p)}(g^p)$ and $h_{qt}^{(p)'}(g^p)$ are given certain scalar coefficients.

Using the relations (3.2) and the subvector $U^{(\mathbf{k}\nu)}\langle p\rangle$, from (6.3) we obtain

$$[\tau_{\mathbf{k}\nu}(e) \times H_p(g^p)] U^{(\mathbf{k}\nu)}\langle q\rangle = [\overline{\tau_{\mathbf{k}\nu}(g^p)} \times H_p'(g^p)] U^{(\mathbf{k}\nu)}\langle p\rangle \tag{6.5}$$

after elementary transformations of the equalities (6.4), written for the components of the vectors $U_{\mathbf{k}\nu\mu}$ $(\mu = 1, 2, \ldots, m_{\mathbf{k}\nu})$.

Here $H_p(g^p)$ and $H_p'(g^p)$ are square matrices of order n_p compiled from the coefficients $h_{qt}^{(p)}(g^p)$ and $h_{qt}^{(p)'}(g^p)$ $(q, t = 1, \ldots, n_p)$, while the symbol $[\tau_{\mathbf{k}\nu}(g) \times H_p(g)]$ denotes the tensor product of the corresponding matrices.

From (6.5), it follows that the subvectors $U^{(\mathbf{k}\nu)}\langle p\rangle$ must satisfy the equations

$$\begin{gathered} \Phi_p^{(\mathbf{k}\nu)}(g^p) U^{(\mathbf{k}\nu)}\langle p\rangle = 0 \\ \Phi_p^{(\mathbf{k}\nu)}(g^p) = [\tau_{\mathbf{k}\nu}(e) \times H_p(g^p)] - [\overline{\tau_{\mathbf{k}\nu}(g^p)} \times H_p'(g^p)] \end{gathered} \tag{6.6}$$

and, consequently, occur in the intersection of the null-spaces of the matrices $\Phi_p^{(\mathbf{k}\nu)}(g^p) \times (g^p \in G^p)$. By the reverse reasoning, we may prove that the components of any vectors in this intersection satisfies equations (6.4). Thus, the space $L^{(\mathbf{k}\nu)}\langle p\rangle$ constitutes the intersection of the null-spaces of the matrices $\Phi_p^{(\mathbf{k}\nu)}(g^p)$ $(g^p \in G^p)$.

The foregoing allow us to, in fact, construct bases in the spaces $L^{(\mathbf{k}\nu)}\langle p\rangle$, $L^{(\mathbf{k}\nu)}$ and $L_{\mathbf{k}\nu\mu}$. Suppose that $R_{\mathbf{k}\nu p}$ is the dimension of the space $L^{(\mathbf{k}\nu)}\langle p\rangle$, and let the vectors $E_\gamma^{(\mathbf{k}\nu)}\langle p\rangle$ $(\gamma = 1, \ldots, R_{\mathbf{k}\nu p})$ form an orthonormalized basis in $L^{(\mathbf{k}\nu)}\langle p\rangle$. We introduce the vectors

$$E_{\gamma p}^{(\mathbf{k}\nu)} \in L^{(\mathbf{k}\nu)}, \quad E_{\mathbf{k}\nu\rho}^{\gamma p} \in L_{\mathbf{k}\nu\rho} \quad (\gamma = 1, 2, \ldots, R_{\mathbf{k}\nu p};\ p = 1, 2, \ldots, N)$$

determined by the equalities

$$E_{\gamma p}^{(\mathbf{k}\nu)}\langle q\rangle = \delta_{pq} E_\gamma^{(\mathbf{k}\nu)}\langle p\rangle, \quad E_{\mathbf{k}\nu\rho}^{\gamma p}\langle q, e\rangle = E_{\gamma p}^{(\mathbf{k}\nu)}\langle \rho, q\rangle \quad (q = 1, 2, \ldots, N) \tag{6.7}$$

where $\delta_{p_{q}}$ is the Kronecker symbol. The systems compiled from such vectors

$$\{\{E_{\gamma p}^{(\mathbf{k}\nu)}\}_{\gamma=1}^{R_{\mathbf{k}\nu p}}\}_{p=1}^{N}, \quad \{\{E_{\mathbf{k}\nu\mu}^{\gamma p}\}_{\gamma=1}^{R_{\mathbf{k}\nu p}}\}_{p=1}^{N} \tag{6.8}$$

constitutes an orthonormalized basis in the space $L^{(\mathbf{k}\nu)}$ and an orthogonal basis in $L_{\mathbf{k}\nu\mu}$, respectively.

Let us show that the scalar product in the space $L_{\mathbf{k}\nu\mu}$, which, unlike the oridinary scalar product (U, V) is denoted by $(U, V)_G$, is given by the equality

$$(U, V)_G = M_G[(U\langle g\rangle, V\langle g\rangle)] \tag{6.9}$$

We will understand by M_G a special functional on G (for finite groups, an averaging functional) introduced previously /8/ and possessing the property

$$M_G[\overline{\tau_{\mathbf{k}\nu\mu\rho}(g)}\,\tau_{\mathbf{k}_1\nu_1\mu_1\rho_1}(g)] = \delta_{\mathbf{k}\mathbf{k}_1}\delta_{\nu\nu_1}\delta_{\rho\rho_1}\delta_{\mu\mu_1}/m_{\mathbf{k}\nu} \tag{6.10}$$

In this scalar product, the norm of the vectors $E_{\mathbf{k}\nu\mu}^{\gamma p}$ is equal to $1/\sqrt{m_{\mathbf{k}\nu}}$. It follows from (6.9), (3.2), (6.7), (6.2) and (6.10) that

$$(BE_{\mathbf{k}\nu\mu}^{\varepsilon q}, E_{\mathbf{k}\nu\mu}^{\gamma p})_G = \frac{1}{m_{\mathbf{k}\nu}}\sum_{g\in G}\sum_{\rho_1,\rho_2=1}^{m_{\mathbf{k}\nu}}\overline{\tau_{\mathbf{k}\nu\rho_1\rho_2}(g)}\,(B\langle p, e, q, g\rangle E_{\varepsilon}^{(\mathbf{k}\nu)}\langle\rho_2, q\rangle, E_{\gamma}^{(\mathbf{k}\nu)}\langle\rho_1, p\rangle) \tag{6.11}$$

Now, using techniques of linear algebra we construct the matrix $B_*^{(\mathbf{k}\nu)}$ corresponding to the operator $B^{(\mathbf{k}\nu)}$ is the basis adopted for space $L^{(\mathbf{k}\nu)}$. By (4.6), it must have the form

$$B_*^{(\mathbf{k}\nu)} = \|B_*^{(\mathbf{k}\nu)}\langle p, q\rangle\|_{p,q=1}^{N} \tag{6.12}$$
$$B_*^{(\mathbf{k}\nu)}\langle p, q\rangle = \|m_{\mathbf{k}\nu}(BE_{\mathbf{k}\nu\mu}^{\varepsilon q}, E_{\mathbf{k}\nu\mu}^{\gamma p})_G\|_{\gamma,\varepsilon=1}^{R_{\mathbf{k}\nu p},\,R_{\mathbf{k}\nu q}}$$

Using expressions (6.11), we finally obtain

$$B_*^{(\mathbf{k}\nu)}\langle p, q\rangle = \|(D_*^{(\mathbf{k}\nu)}\langle p, q\rangle E_{\varepsilon}^{(\mathbf{k}\nu)}\langle q\rangle, \quad E_{\gamma}^{(\mathbf{k}\nu)}\langle p\rangle)\|_{\gamma,\varepsilon=1}^{R_{\mathbf{k}\nu p},\,R_{\mathbf{k}\nu q}} \tag{6.13}$$

$$D_*^{(\mathbf{k}\nu)}\langle p, q\rangle = \sum_{g\in G}[\overline{\tau_{\mathbf{k}\nu}(g)}\times B\langle p, e, q, g\rangle] \tag{6.14}$$

Thus, the operator equation (4.5) reduces to the system of algebraic equations

$$B_*^{(\mathbf{k}\nu)}X^{(\mathbf{k}\nu)} = Y^{(\mathbf{k}\nu)} \tag{6.15}$$
$$X^{(\mathbf{k}\nu)} = \|X^{(\mathbf{k}\nu)}\langle p\rangle\|_{p=1}^{N}, \quad X^{(\mathbf{k}\nu)}\langle p\rangle = \|X^{(\mathbf{k}\nu)}\langle\gamma, p\rangle\|_{\gamma=1}^{R_{\mathbf{k}\nu p}}$$

whose matrix is determined by expression (6.13). In this case

$$Y^{(\mathbf{k}\nu)}\langle p, \gamma\rangle = (V^{(\mathbf{k}\nu)}\langle p\rangle, E_{\gamma}^{(\mathbf{k}\nu)}\langle p\rangle) \tag{6.16}$$
$$U^{(\mathbf{k}\nu)}\langle p\rangle = \sum_{\gamma=1}^{R_{\mathbf{k}\nu p}} X^{(\mathbf{k}\nu)}\langle\gamma, p\rangle E_{\gamma}^{(\mathbf{k}\nu)}\langle p\rangle$$

Example. We construct the resolvent system of algebraic equations for the plane girder of Fig.1, which possesses the symmetry group C_{4v}. Rotations C_m by angles $m\pi/2$ and reflections Θ_m in the planes Π_m $(m = 0, 1, 2, 3)$ are the symmetry elements. In Fig.1 elementary cells of the girder are denoted, the nodes in the cell Ω^e (coinciding with the nodes of the girder) are enumerated, and the directions of the unit load forces V_{051} that occur in the set formed by the representation τ_{05} indicated. The unit forces forming the load V_{052} are shown by the broken lines.

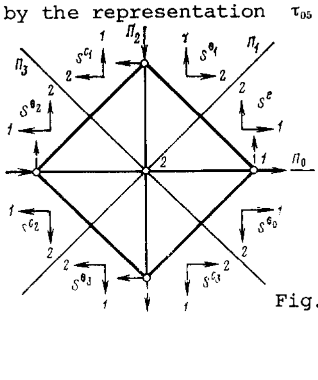

Fig.1

The matrix $\tau_{05}(g)$ has the form

$$\tau_{05}(C_m) = \begin{Vmatrix} c & s \\ -s & c \end{Vmatrix}, \quad \tau_{05}(\Theta_m) = \begin{Vmatrix} c & -s \\ -s & -c \end{Vmatrix}; \quad c = \cos\frac{m\pi}{2}, \quad s = \sin\frac{m\pi}{2}$$

Let us present necessary initial data, denoting by $U\langle t, p, g\rangle$ the motion of the node $gz_p \in S^g$ in the axial direction with number t from the frame of reference ω^g. In the cell Ω^e, the number of nodes $N = 2$. All the components of the subvectors $U\langle 2, g\rangle$ $(g \neq e)$ and $U\langle\Theta_m\rangle$ $(m = 0, 1, 2, 3)$ will be assumed to be redundant. With this in mind, we set

$$B\langle e,e\rangle=\begin{Vmatrix} a_1 & 0 & a_4 & 0\\ 0 & a_2 & 0 & a_5\\ a_4 & 0 & a_3 & 0\\ 0 & a_5 & 0 & a_3\end{Vmatrix},\quad B\langle e,C_1\rangle=\begin{Vmatrix} a_6 & a_7 & 0 & 0\\ -a_7 & a_8 & 0 & 0\\ 0 & -a_5 & 0 & 0\\ a_4 & 0 & 0 & 0\end{Vmatrix},\quad B\langle e,C_2\rangle=\begin{Vmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -a_4 & 0 & 0 & 0\\ 0 & -a_5 & 0 & 0\end{Vmatrix},\quad B\langle e,C_3\rangle=\begin{Vmatrix} a_6 & -a_7 & 0 & 0\\ a_7 & a_8 & 0 & 0\\ 0 & a_5 & 0 & 0\\ -a_4 & 0 & 0 & 0\end{Vmatrix}$$

Here a_i $(i=1,2,\ldots,8)$ are the corresponding rigidity characteristics of the girder. By (6.3) and (6.1), $V^{(05)}=[1,0,0,1,0,0,0,0]$ and $G^1=\{e,\Theta_0\}$ and $G^2=C_{4v}$ are the groups of the nodes z_1 and z_2, respectively. The coefficient matrices compiled from the coupling equations (6.4) and necessary for subsequent computations are written as follows:

$$H_p(e)=\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix},\quad H_1'(\Theta_0)=H_2'(\Theta_0)=\begin{Vmatrix}1 & 0\\ 0 & -1\end{Vmatrix},\quad H_2'(\Theta_1)=\begin{Vmatrix}0 & 1\\ 1 & 0\end{Vmatrix}$$

Now, using (6.6), we find the matrix

$$\Phi_p^{(05)}(\Theta_0)=\operatorname{diag}[0,2,2,0]\quad (p=1,2),\qquad \Phi_2^{(05)}(\Theta_1)=\begin{Vmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\end{Vmatrix}$$

and construct the orthonormalized bases in the spaces $L^{(05)}\langle 1\rangle$ and $L^{(05)}\langle 2\rangle$. Note that the first space is the null-space of $\Phi_1^{(05)}(\Theta_0)$, while the second is the intersection of the null-spaces of the matrices $\Phi_2^{(05)}(\Theta_0)$ and $\Phi_2^{(05)}(\Theta_1)$, so that we have

$$R_{051}=2;\quad E_1^{(05)}\langle 1\rangle=[1,0,0,0],\quad E_2^{(05)}\langle 1\rangle=[0,0,0,1]$$
$$R_{052}=1;\quad E_1^{(05)}\langle 2\rangle=[\sqrt{2}/2,\ 0,\ 0,\ -\sqrt{2}/2]$$

Moreover, it follows from (6.14) that

$$D_*^{(05)}\langle 1,1\rangle=\begin{Vmatrix} a_1 & 0 & 0 & 2a_7\\ 0 & a_2 & -2a_7 & 0\\ 0 & -2a_7 & a_1 & 0\\ 2a_7 & 0 & 0 & a_2\end{Vmatrix},\quad D_*^{(05)}\langle 2,1\rangle=\begin{Vmatrix} 2a_4 & 0 & 0 & -2a_5\\ 0 & 2a_5 & 2a_4 & 0\\ 0 & 2a_5 & 2a_4 & 0\\ -2a_4 & 0 & 0 & 2a_5\end{Vmatrix}$$

$$D_*^{(05)}\langle 1,2\rangle=\operatorname{diag}[a_4,a_5,a_4,a_5],\quad D_*^{(05)}\langle 2,2\rangle=\operatorname{diag}[a_3,a_3,a_3,a_3]$$

Using expressions (6.13) and (6.16), we obtain the required system (6.15):

$$\begin{Vmatrix} a_1 & 2a_7 & a_4/\sqrt{2}\\ 2a_7 & a_2 & -a_5/\sqrt{2}\\ 2\sqrt{2}a_4 & -2\sqrt{2}a_5 & a_3\end{Vmatrix}X^{(05)}=\begin{Vmatrix}1\\1\\0\end{Vmatrix}$$

7. Plane problem for isotropic medium. Generalized periodic problems are the problems in this class that have received the most comprehensive study. They have been solved by the Kosmodamianskii method /9/ and the small parameter method /10/.

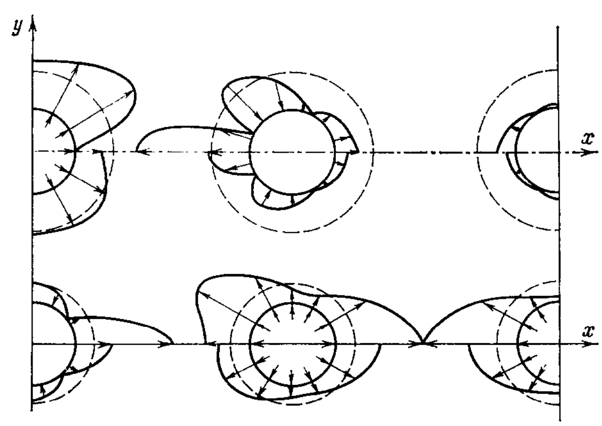

Fig.2

Let us show that the general technique proposed in Sect.5 can be applied. Using previous results /9/ for this technique, the concentration of stresses in a medium weakened by a regular series of circular holes was computed for two cases of loading by an internal pressure of intensity q: (a) only the basic contour is loaded; (b) all contours, other than the basic contour, is loaded. Fig.2 depicts the pathwise stress diagrams σ_θ/q (the unit stress diagram is given by the broken line). The upper part of the Fig.1 refers to case (a), and the lower part, to case (b). Stress diagrams located above (below) the x-axis correspond to a value $d=0.4\ R(d=R)$, with R the radius of the hole and d the thickness of the connecting strip between the holes. Note the two qualitative effects: there also exist values d_0 and d_0' such that if $d<d_0$ and if only a single contour is loaded, its stress concentration is less that of a neighboring (unloaded) contour, while when $d<d_0'$ for the case of a single

unloaded contour, the stress concentration on this contour will be greater than for the ordinary periodic problem.

A numerical analysis of these effects is illustrated by the curves for the concentration coefficient K in Fig.3. Curves *1* and *2* correspond to the basic contour and a neighboring contour under the load condition (a). Curve 3 represents the difference between the concentration coefficients of the stresses on the basic contour for an ordinary periodic problem and for loading (b). According to Fig.3, $d_0 \approx 0.8R$ and $d_0' \approx 0.4R$.

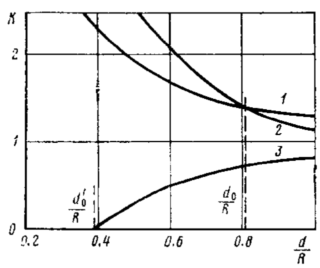

Fig.3

These methods of solving generalized periodic problems may be used for other symmetry groups. With this in mind, the small parameter method is quite obvious.

Expressions for the complex potentials $\varphi_{k\nu\mu}(z)$ and $\psi_{k\nu\mu}(z)$ have been compiled /5/ for a fixed coordinate system. We write them in a new coordinate system with origin at the center of the main hole, which is obtained from the initial system by parallel transfer. By a_r $(r = 1, 2)$ and D, we will understand the complex numbers corresponding to the basic vector $\mathbf{a}_r$ and the transport vector.

Following the basic procedures of the previous method /10/, we expand the functions $\varphi_{k\nu\mu}(z)$ and $\psi_{k\nu\mu}(z)$ in a series in powers of the small parameter $\varepsilon = 1/a_1$, and set

$$\Phi^{(\eta)}(z) = \sum_{s=0}^{\infty} \varepsilon^s \Phi_s^{(\eta)}(z), \quad \Psi^{(\eta)}(z) = \sum_{s=0}^{\infty} \varepsilon^s \Psi_s^{(\eta)}(z)$$

We substitute all these expansions in equation (4.7) and transform them in a series $(s = 0, 1, \ldots)$ of systems

$$K_1\Phi_s^{(\rho)}(t) + K_2[\overline{t\Phi_s^{(\rho)'}(t)} + \overline{\Psi_s^{(\rho)}(t)}] = f_s^{(\rho)}(t) \quad (\rho = 1, 2, \ldots, m_{k\nu}) \tag{7.1}$$

Here

$$f_0^{(\rho)}(t) = f^{(\rho)}(t)$$

$$f_s^{(\rho)}(t) = -K_1 \sum_{r=0}^{s-1} \sum_{p=0}^{r} J_{\Phi sr}^{(\rho p00)} - K_2 \Big\{ \sum_{r=0}^{s-2} \sum_{p=0}^{r} (r - p + 1)\overline{J_{\Phi s, r+1}^{(\rho p00)}} t/\bar{t} +$$

$$\sum_{r=0}^{s-1} \sum_{p=0}^{r} [\overline{J_{\Psi sr}^{(\rho p20)}} + (r + 1)\overline{J_{\Phi sr}^{(\rho p11)}}] \Big\} \quad (s = 1, 2, \ldots)$$

$$J_{\Lambda sr}^{(\rho pqj)} = \sum_{\eta=1}^{m_{k\nu}} C_r^p (-1)^p t^{r-p} [\lambda_{r+j+1, r+q-p-1}^{(\rho\eta)(j+1)} I_{\Lambda sr}^{(\eta p)} + \lambda_{r+j+1, r+q-p-1}^{(\rho\eta)(j+3)} \overline{I_{\Lambda sr}^{(\eta p)}}]$$

$$I_{\Lambda sr}^{(\eta p)} = \frac{1}{2\pi i} \int_{\Gamma} \Lambda_{s-r-1}^{(\eta)}(t) t^p \, dt \quad (\Lambda = \Phi, \Psi)$$

$$\lambda_{rp}^{(\rho\eta)(j+1)} = \lim_{N\to\infty} \sum_{m_1, m_2=-N}^{N} \sum_{m=0}^{n-1}{}^{*} \tau_{k\nu\rho\eta}(T_{m_1m_2}C_m) e^{-imp\alpha} \times$$

$$[1 + j(\bar{\varepsilon}_3 - \bar{\varepsilon}_3 e^{-im\alpha} - m_1\bar{\varepsilon}_1 - m_2\bar{\varepsilon}_2 - 1)][e^{-im\alpha}(m_1\varepsilon_1 + m_2\varepsilon_2) + \varepsilon_3(1 - e^{-im\alpha})]^{-r}$$

$$\lambda_{rp}^{(\rho\eta)(j+3)} = \lim_{N\to\infty} \sum_{m_1, m_2=-N}^{N} \sum_{m=0}^{n-1} \tau_{k\nu\rho\eta}(T_{m_1m_2}C_m\Theta) e^{-imp\alpha} \times$$

$$[1 + j(\bar{\varepsilon}_3 - \varepsilon_3 e^{-im\alpha} - m_1\bar{\varepsilon}_1 - m_2\bar{\varepsilon}_2 - 1)][e^{-im\alpha}(m_1\varepsilon_1 + m_2\varepsilon_2) + \bar{\varepsilon}_3 - \varepsilon_3 e^{-im\alpha}]^{-r}$$

$$\varepsilon_1 = a_1/|a_1|, \quad \varepsilon_2 = a_2/|a_1|, \quad \varepsilon_3 = D/|a_1|, \quad \alpha = 2\pi/n$$

and C_r^p is the number of combinations of r elements taken p at a time, while the index j takes the values 0 and 1, finally the asterisk following the summation sign indicates that terms with subscripts $m_1 = m_2 = m = 0$ are absent.

A solution of the system (7.1) for a fixed value of s may be carried out by the Muskhelishvili method /1/ and determines the s-th approximation of the required functions $\Phi^{(\eta)}(z)$ and $\Psi^{(\eta)}(z)$.

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Translated by R.H.S.